IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

ON THE PROJECTIVE EQUIVALENCE BETWEEN FIRST APPROXIMATE MATSUMOTO METRIC AND KROPINA METRIC

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ABSTRACT
In this article, we find equations to characterize projective equivalence between two special classes of (α, β)-metrics $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\alpha^2}{\beta}$, where α and $\bar{\alpha}$ are two Riemannian metrics and β and $\bar{\beta}$ are two one forms. Further, we study projective relation between same these two (α, β)-metrics.

AMS Classification: 53C60, 53B40.

KEYWORDS: Finsler Space, (α, β)-metrics, Projective equivalence, Douglas metric.

INTRODUCTION
In projective Finsler geometry, we study projectively equivalent Finsler metrics on a manifold, namely, geodesics are same up to a parametrization. J. Douglas introduces two projective quantities: the (projective) Douglas curvature and the (projective) Weyl curvature. The Douglas curvature always vanishes for Riemannian metrics and the Weyl curvature is an extension of the Weyl curvature in Riemannian geometry. Finsler metrics with vanishing Douglas curvature are called Douglas metrics. Roughly speaking, Douglas metrics are locally projectively Finsler metrics. Douglas metrics form an important class of metrics in Finsler geometry. Two Finsler metrics L and $\bar{L}$ on a manifold M are said to be projectively equivalent if they have the same geodesics as point sets.

Two regular metrics on a manifold are said to be pointwise projectively related if they have the same geodesics as point sets. The projective relation is said to be trivial if the corresponding sprays are equal. Two regular metric paces are said to be projectively related if there is a diffeomorphism between them such that the pull-back metric metric is pointwise projective to another one. The projective changes between two Finsler spaces have been studied by many geometers [4] [9] [10] [11].

An (α, β)-metrics form a special and important class of Finsler metrics which can be expressed in the form $L = \alpha \, \phi(s)$, $s = \beta/\alpha$, where $\alpha = \alpha(x, y) = \sqrt{a_{ij}(x, y)y^i y^j}$ is a Riemannian metric, $\beta = \beta(y) = b_i(x)y^i$ is a 1-form on M^n and $\phi(s)$ is a C^∞ positive function on some open interval. In particular, when $\phi(s) = 1/s$, the Finsler metric $\bar{L} = \frac{\alpha^2}{\beta}$ is called a Kropina metric. Kropina metrics were first introduced by L. Berwald in connection with a twodimensional Finsler space with rectilinear extremal and were investigated by V. K. Kropina. They together with Randers metrics are C-reducible. However, Randers metrics are regular Finsler metrics but Kropina metrics are non-regular Finsler metrics.

S. Bácsó et al studied curvature properties of (α, β)-metrics and also studied Finsler spaces of Douglas type [1] [2], X. Cheng and Z. Shen worked On Douglas metrics [3], and also Shen et al studied On a class of Douglas metrics [6].

The purpose of this paper is to study projective equivalence between two (α, β)-metrics, and also we work on projective relation between the two (α, β)-metrics.

PRELIMINARIES

The terminology and notations are referred to [7] [8]. Let $F^n = (M, L)$ be a Finsler space on a differential manifold M endowed with a fundamental function $L(x, y)$. We use the following notions:

a) $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \ \dot{\partial}_i = \frac{\partial}{\partial y^i}$,

b) $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$,

c) $h_{ij} = g_{ij} - l_i l_j$,

d) $\gamma^i_{jk} = \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk})$,

e) $G^i = \frac{1}{2} \gamma^i_{jk} y^j y^k, \ G^i_j = \dot{\partial}_j G^i, G^i_{jk} = \dot{\partial}_k G^i_j, \ \ G^i_{jkl} = \dot{\partial}_l G^i_{jk}$.

The concept of (α, β)-metric $L(\alpha, \beta)$ was introduced in 1972 by M. Matsumoto and studied by many authors. The Finsler space (M, L) is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x, y) y^i y^j$ and $\beta = b_i(x) y^i$.

A change $L \to \bar{L}$ of a Finsler metric on a same underlying manifold M is called projective change if any geodesic in (M, L) remains to be a geodesic in $(M, \bar{L})$ and viceversa. We say that a Finsler metric is projectively related to another metric if they have the same geodesics as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [3]
$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i,$$
Where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^j) denotes the local coordinates in the tangent bundle TM.

Two Finsler metrics L and $\bar{L}$ are projectively related if and only if their spray coefficients have the relation [3]
$$G^i = \bar{G}^i + P(y) y^i,$$
Where $p(y)$ is a scalar function on $TM \setminus \{0\}$ and homogeneous of degree one in y. A Finsler metric is called a *projectively flat* metric if it is projectively related to a locally Minkowskian metric.

For a given Finsler metric $L = L(x, y)$, the geodesic of L satisfy the following system of differential equation:
$$\frac{d^2 x^i}{dt^2} + 2G^i \left(x, \frac{dx}{dt}\right) = 0,$$
where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by,
$$G^i = \frac{1}{2} g^{ij} \{(L^2)_{x^m y^l} y^m - (L^2)_{x^l}\}.$$

Two Finsler metrics L and $\bar{L}$ on a manifold M are said to be (pointwise) projectively related if they have the same geodesics as point sets. The equivalent condition has been characterized using spray coefficients.
By the definition, an (α, β)-metric is a Finsler metric expressed in the following form,
$$L = \alpha \phi(s), \qquad s = \frac{\beta}{\alpha},$$
where α is a Riemannian metric and β is a 1-form with $\|\beta_x\| < b_0, \ x \epsilon M$. The function $\phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying
$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

For Kropina metric $\bar{L} = \frac{\alpha^2}{\beta}$, it is easy to see that it is not a regular (α, β)-metric but the relation $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ is still true for $|s| > 0$. In this case, the fundamental form of the metric tensor induced by L is positive definite.
Let
$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \qquad s_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad r_i - r_{ij} b^j,$$

where $b_{i;j}$ means the coefficients of the covariant derivative of β with respect to α. and put $r_{00} = r_{ij}y^i y^j$, $r_0 = r_i y^j$, $s_{l0} = s_{li}y^i$, $s_0 = b^l s_{l0}$, etc.

The geodesic coefficients Gi of F and geodesic coefficients G_α^i of α are related as follows

$$G^i = G_\alpha^i + \alpha Q s_0^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + \psi\{-2\alpha Q s_0 + r_{00}\}b^i, \tag{2.1}$$

where, $s_j^i = a^{ik}s_{kj}$ and

$Q = \frac{\phi'}{\phi - s\phi''}$,

$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}$,

$\psi = \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')}$.

Definition 2.1. *The tensor* $D = D_{jkl}^i \partial_i \otimes dx^i \otimes dx^k \otimes dx^l$, *where*

$$D_{jkl}^i = \frac{\partial^3}{\partial y^i \partial y^k \partial^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right)$$

is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

One can see that the Douglas tensor is a projective invariant and it is a non-Riemannian quantity i.e., it is vanishes for Riemannian metrics.

Ningwei Cui and Yi-Bing Shen in [10], defined the Douglas tensor of a general (α, β)-metric as

$$D_{jkl}^i = \frac{\partial^3}{\partial y^i \partial y^k \partial^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right) \tag{2.2}$$

where

$$T^i = \alpha Q s_0^i + \psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.3}$$

and

$$T_{y^m}^m = Q' s_0 + \psi' \alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \tag{2.4}$$

In the sequel, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{L}$.

Let L and $\bar{L}$ be two (α, β)-metrics, we assume that they have same Douglas tensor i.e., $D_{jkl}^i = \bar{D}_{jkl}^i$. From (2.2), we have

$$\frac{\partial^3}{\partial y^i \partial y^k \partial^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}(T_{y^m}^m - \bar{T}_{y^m}^m)\right)y^i = 0.$$

Then there exists a class of scalar functions $H_{jk}^i = H_{jk}^i(x)$, such that

$$\left(T^i - \bar{T}^i - \frac{1}{n+1}(T_{y^m}^m - \bar{T}_{y^m}^m)\right)y^i = H_{00}^i, \tag{2.5}$$

where $H_{00}^i = H_{jk}^i(x)y^j y^k$, T^i and $T_{y^m}^m$ ym are given by (2.3) and (2.4) respectively.

PROJECTIVE EQUIVALENCE BETWEEN TWO (α, β)-METRICS

In this secttion, we find he projective equivalence between First approximate Matsumoto metric and Kropina metric, that is, $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a same underlying manifold M of dimension $N \geq 3$. For (α, β)-metric $L = \alpha + \beta + \frac{\beta^2}{\alpha}$, L is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\| < 1$, for any $x \epsilon M$. the geodesic coefficients are given by (2.1) with

$Q = \frac{1+2s}{1-s^2}$,

$\Theta = \frac{1 - 3s^2 - 4s^3}{2(1+s+s^2)(1-3s^2+2b^2)}$,

$\psi = \frac{1}{1-3s^2+2b^2}$. $\tag{3.1}$

For Kropina metric $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$, the geodesic coefficients are given by (2.1) with

$Q = \frac{1}{2s}$,

$\Theta = \frac{s}{b^2}$,

$\psi = \frac{1}{2b^2}$. $\tag{3.2}$

Lemma 3.1. *Consider the two (α, β)-metrics, that is, First approximate Matsumoto metric $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and Kropina metric $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a manifold M with dimension $n \geq 2$. Then they have the same Douglas tensor if and only if both the metrics L and $\bar{L}$ are Douglas metrics.*

Proof: Firstly, we prove the sufficient condition. Let L and $\bar{L}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$, i.e., both L and $\bar{L}$ have same Douglas tensor. Next, we prove the necessary condition. If L and $\bar{L}$ have the same Douglas tensor, then (2.5) holds. Substituting (3.1) and (3.2) into (2.5), we get the following,

$$\frac{A^i\alpha^9 + B\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + J^i}{K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} = H^i_{00},$$

(3.3)

$A^i = (1 + 2b^2)\{s^i_0(1 + 2b^2) - 2s_0 b^i\}$,
$B^i = (1 + 2b^2)\{2\beta\{s^i_0(1 + 2b^2) - 2s_0 b^i\} + r_{00}b^i - 2\lambda y^i(s_0 + r_0)\}$,
$C^i = -\beta\left\{s^i_0\{7 + 4b^2(4 + b^2)\} - 4s_0\{\beta b^i(2 + b^2) + 3\lambda y^i b^2\}\right\}$,
$D^i = \beta\{2\beta\{-(1 + 2b^2)(7 + 2b^2)s^i_0\beta + \lambda y^i\{5(s_0 + r_0) + 4b^2(4s_0 + r_0)\} + 4s_0 b^i\beta(b^2 + 2)\} - r_{00}\{b^i(5 +$
$\quad 4b^2)\beta + 6b^2\lambda y^i\}\}$,
$E^i = 3\beta^3\{\beta\{s^i_0(5 + 4b^2) - 2s_0 b^i\} - 4s_0(b^2 + 1)\lambda y^i\}$,
$F^i = \beta^3\left\{6\beta^2\{(5 + 4b^2)s^i_0 - 2s_0 b^i\} + b^i\beta(7 + 2b^2)r_{00} - 2\lambda y^i\{r_0(7 + 2b^2)\beta - 3r_{00}(2b^2 + 1) + (19 +$
$\quad 14b^2)\beta s_0\}\right\}$,
$G^i = 3\beta^5\{4s_0\lambda y^i - 3s^i_0\beta\}$,
$H^i = -3\beta^5\{\beta(5s^i_0\beta + r_{00}b^i) - 2\lambda y^i\{(b^2 + 2)r_{00} - \beta(5s_0 + r_0)\}\}$,
$J^i = 6r_{00}\beta^7\lambda y^i$,

(3.4)

and

$K = (1 + 2b^2)^2$,
$L = -4\beta^2(1 + 2b^2)(2 + b^2)$,
$M = \beta^4\{(1 + 2b^2)^2 + 3(8b^2 + 7)\}$,
$N = -12\beta^6(b^2 + 2)$,
$P = 9\beta^8$,

(3.5)

and also,

$\bar{A}^i = (\bar{b}^2\bar{s}^i_0 - \bar{s}_0\bar{b}^i)$,
$\bar{B}^i = \bar{\beta}\{2\lambda y^i(\bar{r}_0 + \bar{s}_0) - \bar{r}_{00}\bar{b}^i\}$.

(3.6)

Here $\lambda = \frac{1}{n+1}$. Further, (3.3) is equivalent to,

$\{A^i\alpha^9 + B\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + J^i\}(2\bar{b}^2\bar{\beta})$
$\qquad + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P)$
$\qquad = H^i_{00}(2\bar{b}^2\bar{\beta})(K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P).$

(3.7)

Replacing y^i by $-y^i$ in (3.7), we get

$\{-A^i\alpha^9 + B\alpha^8 - C^i\alpha^7 + D^i\alpha^6 - E^i\alpha^5 + F^i\alpha^4 - G^i\alpha^3 + H^i\alpha^2 + J^i\}(-2\bar{b}^2\bar{\beta})$
$\qquad - (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P)$
$\qquad = H^i_{00}(-2\bar{b}^2\bar{\beta})(K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P).$

(3.8)

Addition of equation (3.7) and (3.8) we have,

$\{A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3\}(\bar{b}^2\bar{\beta}) = 0.$

(3.9)

From equation (3.9), we obtain

$A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3 = 0.$

Thus, we conclude that (3.3) is equivalent to,

$$\frac{B\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + J^i}{K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} = H_{00}^i$$

(3.10)

equation (3.10) is similar to

$$\{B\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + J^i\}(2\bar{b}^2\bar{\beta}) + \{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i\}\{K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P\}$$
$$= H_{00}^i(2\bar{b}^2\bar{\beta})\{K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P\}$$

From this, we observed that $\bar{A}^i\bar{\alpha}^2\{K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P\}$ can be devided by $\bar{\beta}$. Since $\beta = \mu\bar{\beta}$ then $\bar{A}^i\bar{\alpha}^2K\alpha^8$ can be devided by $\bar{\beta}$ is prime withrespect to α and $\bar{\alpha}$, thus, $\bar{A}^i = \bar{b}^2\bar{s}_0^i - \bar{b}^2\bar{s}_0$ can be devided by $\bar{\beta}$. Hence, there is a scalar function $\varphi^i(x)$ such that

$$\bar{b}^2\bar{s}_0^i - \bar{b}^2\bar{s}_0 = \bar{\beta}\varphi^i.$$

(3.11)

Contracting (3.11) by $\bar{y}^i = \bar{a}_{ij}y^j$, we have $\varphi^i(x) = -\bar{s}^i$. So we get

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i)$$

(3.12)

Thus, By Lemma 2.2 [4], $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric. Since L and $\bar{L}$ have the same Douglas tensor, both of them are Douglas metrics. Now we prove the main result:

Theorem 3.1. *Let* $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ *and* $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ *be two* (α, β)*-metrics on an n-dimensional manifold* $M(n \geq 3)$, *where* α *and* $\bar{\alpha}$ *are two Riemannian metrics,* β *and* $\bar{\beta}$ *are two one-forms. Then* L *is projectively equivalent to* $\bar{L}$ *if and only if they are Douglas metrics and the geodesic coefficients of* α *and* $\bar{\alpha}$ *have the following relation:*

$$G_\alpha^i + \tau b^i\alpha^2 = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i,$$

where $b^i = a^{ij}b_j$, $\bar{b}^i = \bar{a}^{ij}\bar{b}_j$, $\bar{b}^2 = \|\bar{\beta}\|_{\bar{\alpha}}^2$ *and* $\tau = \tau(x)$ *is a scalar function and* $\theta = \theta_iy^i$, $\xi = \xi_iy^i$ *are 1-form on M.*

Proof: First we prove the necessary condition. If L is projectively eqivalent to $\bar{L}$, they have the same Douglas tensor. By Lemma 3.1, we know that L and $\bar{L}$ are both Douglas metrics. We know that by [9] First approximate Matsumoto metric is a Douglas metric if and only if

$$b_{i;j} = \tau\{(1 + 2b^2)a_{ij} - 3b_ib_j\},$$

(3.13)

where, $\tau = \tau(x)$ is a scalar function on M. In this case, β is closed. Plugging (3.13) and (3.1) into (2.1) we get,

$$G^i = G_\alpha^i + \left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}\tau y^i + \tau b^i\alpha^2.$$

(3.14)

On the other hand, Substitute (3.12) and (3.2) into (2.1) which gives,

$$\bar{G}^i = \bar{G}_\alpha^i - \frac{1}{2\bar{b}^2}\left\{-\bar{\alpha}^2\bar{s}^i + (2\bar{s}_0y^i - \bar{r}_{00}\bar{b}^i) + \frac{2\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\right\}.$$

(3.15)

By the projective equivalence of L and $\bar{L}$ again, there is a scalar function $P = P(x, y)$ on $TM\backslash\{0\}$ such that

$$G^i = \bar{G}^i + Py^i.$$

(3.16)

From (3.14), (3.15) and (3.16) we obtain,

$$\left\{P - \left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}\tau - \frac{1}{\bar{b}^2}\left\{\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right\}\right\}y^i$$
$$= G_\alpha^i - \bar{G}_\alpha^i + \tau b^i\alpha^2 - \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) .$$

(3.17)

Note, the RHS of (3.17) is a quadratic in y. Then there exists a 1-form $\theta = \theta_i(x)y^i$ on M such that

$$\left\{P - \left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}\tau - \frac{1}{\bar{b}^2}\left\{\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right\}\right\}y^i = \theta.$$

Thus, we get

$$G_\alpha^i + \tau b^i\alpha^2 = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i.$$

(3.18)

Thus, completes the proof of necessity. Conversely, By (3.14), (3.15) and (3.18) we obtain，

$$G^i = \bar{G}^i + \left\{\theta + \left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}\tau - \frac{1}{\bar{b}^2}\left\{\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right\}\right\}y^i.$$

(3.19)

Hence L is projective equivalent to $\bar{L}$. This completes the proof of the Theorem.
We obtain immidiately from the above Theorem 3.1

Proposition 1. Let $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be two (α, β)-metrics on an n-dimensional manifold $M(n \geq 3)$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two one-forms. Then L is projectively equivalent to $\bar{L}$ if and only if the following equations holds

$$G_{\alpha}^i + \tau b^i \alpha^2 = \bar{G}_{\alpha}^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i,$$

$$b_{i;j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\},$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i),$$

where $b_{i;j}$ denote the coefficient of the covariant derivatives of β with respect to α.

PROJECTIVELY RELATED TWO (α, β)-METRICS

In this section, we study First approximate Matsumoto metrics $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and Kropina metric $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are projectively related.

Theorem 4.1. Let $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be two (α, β)-metrics on an n-dimensional manifold $M(n \geq 3)$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two one-forms. Then L is projectively related to $\bar{L}$ if and only if
   1. α is projectively related to $\bar{\alpha}$.
   2. $b_{i;j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\}$,
   3. $\bar{\beta}$ is closed.
Proof: Firstly we show that necessary condition.
Since Douglas metric is an invariant under change between two Finsler metrics. If L is projectively related to $\bar{L}$, then they have same Douglas tensor. From above third section we get that L and $\bar{L}$ are Douglas metrics. We know that $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ is a Douglas metrics if and only if

$$b_{i;j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\}. \tag{4.1}$$

Plugging (4.1) into (2.1) with (3.1) yields

$$G^i = G_{\alpha}^i + \tau\left\{\left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}y^i\right\} + \tau b^i \alpha^2.$$

(4.2)
Since β is closed, $s_{ij} = 0$. Thus $s_0^i = 0$ and $s_0 = 0$.
So, from (2.1) and (3.2) we have,

$$\bar{G}^i = \bar{G}_{\alpha}^i + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2 \bar{b}^2}y^i + \frac{r_{00}b^i}{2\bar{b}^2}. \tag{4.3}$$

Since, L is projectively related to $\bar{L}$, there is a function $P = P(y)$ on $TM^n\backslash\{0\}$ such that
$$G^i = \bar{G}^i + Py^i. \tag{4.4}$$
From (4.2), (4.3) and (4.4) we have,

$$G_{\alpha}^i - \bar{G}_{\alpha}^i = \left\{\frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2 \bar{b}^2} - \left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}\tau + P\right\}y^i + \left\{\frac{r_{00}}{2\bar{b}^2} - \tau\alpha^2\right\}b^i. \tag{4.5}$$

The LHS of the above equation is a quadratic form then, there exists a one form $\theta = \theta_i y^i$
and $\xi = \xi_i y^i$ on M such that,

$$\frac{r_{00}\beta}{\alpha^2\bar{b}^2} - \left\{\frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{2(\alpha^2 + \alpha + \beta + \beta^2)}\right\}\tau + P = \theta,$$

$$\frac{r_{00}}{2\bar{b}^2} - \tau\alpha^2 = \xi. \tag{4.6}$$

Then we have,

$$G_{\alpha}^i = \bar{G}_{\alpha}^i + \theta y^i + \xi b^i. \tag{4.7}$$

This completes the proof of necessity. For the sufficiency, we notice that $\bar{\beta}$ is closed. It suffice to prove that α is projectively related to $\bar{\alpha}$. From (4.2), (4.3) and (4.7) we have,

$$G^i = \bar{G}^i_\alpha + \theta y^i + \xi\, b^i. \tag{4.8}$$

i.e., α is projectively related to $\bar{\alpha}$.

CONCLUSION

Two Finsler metrics L and $\bar{L}$ are projectively equivalent if and only if their geodesic coefficients have the following relation:

$$G^i = \bar{G}^i + P(x,y)y^i,$$

where $P(x,y)$ is a scalar function on $TM\backslash\{0\}$ with $P(x,\ \lambda y)\ =\ \lambda P(x,y),\ \lambda > 0$.

In Finsler geometry, there are several important classes of Finsler metrics. Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. The study of Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities.

In the present paper, we proved projective equivalence and relation between First approximate Matsumoto metric $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ and Kropina metric $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$.

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